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A quadratic lower bound for Topswops

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ABSTRACT

A quadratic lower bound for the *topswops* function is exhibited. This provides a non-trivial lower bound for a problem posed by J.H. Conway, D.E. Knuth, M. Gardner and others. We describe an infinite family of permutations, each taking a linear number of steps for the topswops process to terminate, and a chaining process that creates from them an infinite family of permutations taking a quadratic number of steps to reach a fixed point with the identity permutation.

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1. Introduction

Berman et al. [1,8] described the following problem, which they called *reverse card shuffle*. It was originally proposed by J.H. Conway of the University of Cambridge as one of a series of card games and problems based on permuting a set of elements by reversing their order according to various rules, as described in [6]. Conway called the problem *topswops*.

> *A deck of cards is numbered 1 to n in random order. Perform the following operations on the deck. Whatever the number on the top card is, count down that many in the deck and turn the whole block over on top of the remaining cards. Then, whatever the number of the (new) top card, count down that many cards in the deck and turn this whole block over on top of the remaining cards. Repeat the process. Show that the number 1 will eventually reach the top.*

One can view the deck of cards as a permutation on $\{1, 2, \ldots, n\}$. Suppose the card numbered k is the top card. Then, turning over a block of k cards at the top of the deck is modeled by a prefix reversal of size k in the corresponding permutation. The topswops problem was also discussed by Martin Gardner [6], who described other problems defined by Conway, such as *topdrops*, *botdrops*, and *botswops*, which differ from topswops in that (1) in *topdrops* the k cards at the top of the deck are reversed and added to the bottom, (2) in *botswops* and *botdrops* the card at the bottom of the deck determines the number of cards at the top of the deck to be reversed and then added to the top (bottom, respectively) of the deck. Gardner [6] also noted:

> *Let it not be supposed that those Conway card games are trivial. They deal with the theory of set permutations and not only may provide deep theorems but also may have a bearing on practical problems that arise in seemingly unrelated fields.*

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Andy Pepperdine [11] considered efficient algorithms for computing the exact number of steps needed for topswops for small integers n, and Knuth, [9], p. 119, described improvements on Pepperdine's procedures. The online encyclopedia of integer sequences lists the sequence of known exact values for $n \le 16$ [12]. The topswops problem has also been called the *deterministic pancake problem* [13], to distinguish it from the *pancake problem* [2,3,5,7] of sorting a permutation with the minimum number of prefix reversals, where the size of each prefix reversal is arbitrary. Topswops was also considered by S. Elizalde and P. Winkler in their paper [4] on *homing sequences*, i.e. sequences where each move puts an element of a permutation into a designated position with shifting to make room for the change.

Gardner noted in [6] that Herbert S. Wilf at the University of Pennsylvania reported:

> *…a delightful discovery about topswops that provides a proof of the game's finiteness. A card is in its "natural position" if its value is the same as its position. For example, if we have 7, 2, 11, 8, 5, 13, 6, 1, 9, 10, 3, 12, 4 [with face cards changed to corresponding numbers], there are five numbers [cards] (*2, 5, 9, 10, 12*) in natural position. If we take these values as powers of two, we can create what I shall call the Wilf number:* $2^2 + 2^5 + 2^9 + 2^{10} + 2^{12} = 5668$. *After any move in topswops the Wilf number must increase.*
>
> *"The reason that the number increases," Wilf writes, "is that the cards which are in natural position and which were too far … to be reached by the reversal operation will still be in natural position afterward. The fate of numbers [cards] which are involved in the reversal is less clear, except for one thing: the first number [card which was on top] before the move will be in natural position after the move, and its power of two is large enough to drown out any changes from other numbers in the reversal [cards above it] . . .*
>
> *Since the numbers increase steadily but cannot exceed 16,382 [in this example], it follows that the game must halt after at most that many moves. A slightly more careful study shows, in fact, that for a game with n numbers [cards], no more than* 2^{n-1} *moves can take place."*

Gardner [6] writes, "This raises an interesting unsolved question: What arrangement of the … numbers [cards] provides the longest game of topswops?"

Knuth (in a solution on page 119 of problem 108, page 74, of [9]) showed that $F_{n+1} - 1$, where F_{n+1} is the $n+1$st Fibonnaci number, is an upper bound . He conjectured that at least $n \log n$ steps are needed (solution of problem 109, [9] p. 120). We instead show a quadratic lower bound for the topswops process.

We now develop notation for the proofs that follow. Let S_n denote the symmetric group of permutations on $\{1, 2, \ldots, n\}$. Let π and π^1 be permutations in S_n. We define the *topswops function* $f : S_n \to S_n$ by $f(\pi) = \pi^1$ where, if $\pi(1) = j$, then

$$\pi'(i) = \begin{cases} \pi(j - i + 1), & \text{if } 1 \le i \le j, \\ \pi(i), & \text{if } i \ge j + 1. \end{cases}$$

The topswops problem considers for a given permutation π, the *iterates of* f, namely, the sequence $\pi, f(\pi) = \pi^1, f(f(\pi)) = \pi^2, \pi^3, \ldots$. Wilf's argument shows that for every permutation this sequence has a fixed point. That is, there exists an integer i such that $f(\pi^i) = \pi^i$. This happens when and only when π^i is a permutation that has the symbol 1 in the first position.

We will typically write a permutation $\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$ as the sequence $\boldsymbol{\pi}(\mathbf{1}), \boldsymbol{\pi}(\mathbf{2}), \ldots, \boldsymbol{\pi}(\mathbf{n})$ in bold face type when context requires that some or all of the symbols be explicitly listed. Hence, $\boldsymbol{\pi}(\mathbf{i})$ denotes the ith symbol in the permutation π, whereas π^i denotes the ith iterate of the topswops function f.

For any permutation π, let *run*(π) denote the sequence of iterates of f, up to but not including the fixed point. We call run(π) the *run sequence of* π. Let $|\text{run}(\pi)|$ denote the length of this sequence. Let *front*(*run*(π)) denote the derived sequence of first elements from each successive permutation in run(π). That is, front(run(π)) describes the sequence of flips (or prefix reversals) executed by the iteration of the topswops function on input π. Note that $|\text{front}(\text{run}(\pi))| = |\text{run}(\pi)|$.

Let $h : N \to N$ be defined by $h(n) = \max_{\pi \in S_n} \{|\text{run}(\pi)|\}$. That is, $h(n)$ is the length of the longest run sequence of iterates of f over all permutations of length n. Let $h^* : N \to N$ be the related function such that, for any natural number n, $h^*(n)$ is the length of the longest sequence of iterates of f over all permutations *for which the fixed point is the identity permutation.* (Only certain permutations reach the identity permutation under iteration.) Clearly $h(n) \ge h^*(n)$.

Knuth in problem 109, [9], p. 74, asks for good upper and lower bounds on the function $h(n)$. Knuth states that "*$h(n)$ probably grows at least as fast as n* $\log n$ *(by comparison with coupon collecting)*" ([9], p. 120). We prove that $h^*(n)$, and hence also $h(n)$, grows at least as fast as $d \cdot n^2$, for some constant $d > 0$.

2. A quadratic lower bound for $h(n)$

To prove our lower bound, we will describe an infinite family of permutations $\Pi = \{\pi_n\}_{n \in I}$, such that, for all $n \in I$, π_n is a permutation on the integers $\{1, \ldots, n\}$, and, for some constant $d > 0$, $|\text{run}(\pi_n)| \ge d \cdot n^2$. For each natural number $k > 1$, let $\Pi^{(k)}$ denote the infinite family of permutations containing, for $n > k$, all permutations π on $\{1, \ldots, n\}$ such that $\pi(j) = j$, for all $2 \le j \le n - k$. These permutations differ from the identity permutation in at most $k + 1$ positions, namely position 1 and the last k positions.

For reasons that will shortly become clear we are particularly interested in finding permutations in $\Pi^{(k)}$ whose fixed point is the identity permutation. Such a family is $S = \{\sigma_n\}$ in $\Pi^{(8)}$. The permutations σ_n, for $n > 17$, are defined by

$$\sigma_n = \mathbf{n}, (\mathbf{2}, \mathbf{3}, \ldots, \mathbf{n-8}), \mathbf{n-5}, \mathbf{n-6}, \mathbf{n-2}, \mathbf{n-7}, \mathbf{1}, \mathbf{n-3}, \mathbf{n-1}, \mathbf{n-4}.$$

That is, for arbitrary integers n, the symbols $2, \ldots, n-8$ are in place, i.e. in their *natural position* as defined by Wilf [6], and the remaining nine symbols are arranged in the pattern indicated. Furthermore, for infinitely many n, the length of the run sequence for σ_n is at least $c \cdot n$, for some constant $c > 0$. An example of a permutation in S is $\sigma_{26} = \mathbf{26}, (\mathbf{2}, \mathbf{3}, \ldots, \mathbf{18}), \mathbf{21}, \mathbf{20}, \mathbf{24}, \mathbf{19}, \mathbf{1}, \mathbf{23}, \mathbf{25}, \mathbf{22}$.

Lemma 1. *For all* $n \geq 24$, $|\mathrm{run}(\sigma_n)| \geq n/5$.

Proof. For σ_n, the first two flips are n and $n-4$ resulting in the permutation $[\mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \ldots, \mathbf{n-8}], \mathbf{n-5}, \mathbf{n-6}, \mathbf{n-2}, \mathbf{n-7}, \mathbf{1}, \mathbf{n-3}, \mathbf{n-1}, \mathbf{n-4}, \mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{n}$. Observe that the first $n-12$ symbols of this permutation are $n-12$ consecutive integers, namely $5, 6, 7, 8, 9, \ldots, (n-8)$. The next flip will be a flip of size 5 which brings 9 to the front, followed by a flip of size 9 which brings 13 to the front, and so on. So, the next sequence of flips is $5, 9, 13, 17, \ldots$. More precisely, the flip sequence is $\{5+4j \mid \text{for } 0 \leq j \leq (n-14)/4\}$. The length of the flip sequence is therefore at least $3 + (n-10)/4$, which is greater than or equal to $n/5$, for all $n \geq 5$. (Note that the 3 added to the term $(n-10)/4$ in the previous statement is justified by the fact that the full flip sequence has two flips before the sequence $5, 9, 13, 17, \ldots$ starts and continues with additional flips afterward.) □

Lemma 1 shows that the family of permutations $\{\sigma_n\}$ is such that $|\mathrm{run}(\sigma_n)|$ is $\Omega(n)$. Recall that our objective is to describe an infinite family Π of permutations $\{\pi_n\}_{n\in I}$, such that, for some constant $d > 0$, and for all $n \in I$, π_n is a permutation on the integers $\{1, \ldots, n\}$, and $|\mathrm{run}(\pi_n)| > d \cdot n^2$. We will define a chaining technique to convert permutations in S into a new family of permutations that satisfy this objective. Our technique requires that, for each $\sigma_n \in S$, $\mathrm{run}(\sigma_n)$ ends with the identity permutation. This is stated in Lemma 2, but its somewhat lengthy proof will be deferred to the end of this section to allow the reader to absorb the big picture before delving into such technical details.

Lemma 2. *For all* $n \geq 18$, *such that* $n \equiv 2 \pmod 8$, $\mathrm{run}(\sigma_n)$ *ends with the identity permutation.*

We now describe the chaining of permutations in the family $\{\sigma_n \mid n \geq 18 \text{ and } n \equiv 2 \pmod 8\}$ to create a family of permutations Π whose sequence of iterates has length at least $d \cdot n^2$, for some constant $d > 0$.

For a permutation ρ_n in $\Pi^{(t)}$ and a permutation ρ_{n+k} in $\Pi^{(k)}$, define the permutation $\rho_n \oplus \rho_{n+k}$ in $\Pi^{(t+k)}$ by

$$(\rho_n \oplus \rho_{n+k})(i) = \begin{cases} \rho_n(i), & \text{if } 1 \leq i \leq n \text{ and } \rho_n(i) \neq 1, \\ \rho_{n+k}(1), & \text{if } \rho_n(i) = 1, \\ \rho_{n+k}(i), & \text{if } n+1 \leq i \leq n+k. \end{cases}$$

For example, consider $\sigma_{26} = \mathbf{26}, (\mathbf{2}, \mathbf{3}, \ldots, \mathbf{18}), \mathbf{21}, \mathbf{20}, \mathbf{24}, \mathbf{19}, \mathbf{1}, \mathbf{23}, \mathbf{25}, \mathbf{22}$ and $\sigma_{34} = \mathbf{34}, (\mathbf{2}, \mathbf{3}, \ldots, \mathbf{26}), \mathbf{29}, \mathbf{28}, \mathbf{32}, \mathbf{27}, \mathbf{1}, \mathbf{31}, \mathbf{33}, \mathbf{30}$. The permutation $\sigma_{26} \oplus \sigma_{34}$ is as follows: $\mathbf{26}, (\mathbf{2}, \mathbf{3}, \ldots, \mathbf{18}), \mathbf{21}, \mathbf{20}, \mathbf{24}, \mathbf{19}, \mathbf{34}, \mathbf{23}, \mathbf{25}, \mathbf{22}, \mathbf{29}, \mathbf{28}, \mathbf{32}, \mathbf{27}, \mathbf{1}, \mathbf{31}, \mathbf{33}, \mathbf{30}$.

Notice that the first 26 symbols of $\sigma_{26} \oplus \sigma_{34}$ are the same as the first 26 of σ_{26}, except that the symbol 1 is replaced by the symbol 34. The remaining eight symbols of $\sigma_{26} \oplus \sigma_{34}$ are the same as the last eight symbols of σ_{34}.

Lemma 3. *For any* $t > 0$ *and any permutations* ρ_n *in* $\Pi^{(t)}$ *and* ρ_{n+k} *in* $\Pi^{(k)}$, *such that* ρ_n *terminates with the identity permutation,* $\rho_n \oplus \rho_{n+k}$ *is a permutation on* $n+k$ *symbols in* $\Pi^{(t+k)}$ *such that* $|\mathrm{run}(\rho_n \oplus \rho_{n+k})| = |\mathrm{run}(\rho_n)| + |\mathrm{run}(\rho_{n+k})|$.

Proof. It follows easily from the definition of $\oplus$ that $\rho_n \oplus \rho_{n+k}$ is a permutation on $n+k$ symbols and has the integers $2, 3, \ldots, (n-t) = (n+k) - (t+k)$ fixed in their respective positions. Thus, $\rho_n \oplus \rho_{n+k}$ is in $\Pi^{(t+k)}$.

By definition $\rho_n \oplus \rho_{n+k}$ and ρ_n have the same symbols in positions 1 through n, except for one position. This is the position i such that $\rho_n(i) = 1$. Here, $\rho_n \oplus \rho_{n+k}(i) = \rho_{n+k}(1)$. It follows that, if $|\mathrm{run}(\rho_n)| = L$, then the first L elements of $\mathrm{front}(\mathrm{run}(\rho_n \oplus \rho_{n+k}))$ is the same sequence as $\mathrm{front}(\mathrm{run}(\rho_n))$. That is, the two permutations execute the same initial sequence of L flips, until $\mathrm{run}(\rho_n)$ terminates with the identity permutation and the permutation produced after the first L steps of $\mathrm{run}(\rho_n \oplus \rho_{n+k})$ is ρ_{n+k}. (To see this consider the following. Let $\mathrm{run}(\rho_n \oplus \rho_{n+k})|_n$ denote the sequence obtained by restricting each permutation in $\mathrm{run}(\rho_n \oplus \rho_{n+k})$ to its first n elements, and replacing the symbol $(\rho_n \oplus \rho_{n+k})(1)$, wherever it occurs in the restricted permutations, with the symbol 1. Then, $\mathrm{run}(\rho_n \oplus \rho_{n+k})|_n$ is easily seen to be exactly the same sequence of permutations as the sequence $\mathrm{run}(\rho_n)$.) As the Lth element of $\mathrm{run}(\rho_n \oplus \rho_{n+k})$ is the permutation ρ_{n+k}, it follows that $|\mathrm{run}(\rho_n \oplus \rho_{n+k})| = |\mathrm{front}(\mathrm{run}(\rho_n \oplus \rho_{n+k}))| = L + |\mathrm{run}(\rho_{n+k})| = |\mathrm{run}(\rho_n)| + |\mathrm{run}(\rho_{n+k})|$. □

One can, of course, chain together more than two permutations. For permutations $\sigma_n, \sigma_{n+k}, \sigma_{n+2k}, \ldots, \sigma_{n+mk}$, for $m \geq 1$, let $\sigma_n \oplus \sigma_{n+k} \oplus \sigma_{n+2k} \cdots \oplus \sigma_{n+mk}$ denote the permutation $(\ldots((\sigma_n \oplus \sigma_{n+k}) \oplus \sigma_{n+2k}) \ldots \oplus \sigma_{n+mk})$.

Corollary 4. *For all* $m \geq 1$, $\pi_{26+8m} = \sigma_{26} \oplus \sigma_{34} \oplus \sigma_{42} \cdots \oplus \sigma_{26+8m}$ *is a permutation on* $26+8m$ *symbols with* $|\mathrm{run}(\pi_{26+8m})| \geq 4/5m^2 + 6m + 26/5$.

Proof. Using Lemma 3 inductively, $|\mathrm{run}(\pi_{26+8m})| = \sum_{i=0}^{m} |\mathrm{run}(\sigma_{26+8i})|$. By Lemma 1, $|\mathrm{run}(\sigma_{26+8i})| \geq (26+8i)/5$, for all $i \geq 0$. Thus, $|\mathrm{run}(\pi_{26+8m})| \geq \sum_{i=0}^{m} (26+8i)/5 = 4/5m^2 + 6m + 26/5$. □

This establishes the following theorem.

Theorem 5. $h^*(n) = \Omega(n^2)$.

In order to prove Lemma 2, we first make the following observations about various sequences of permutations, where for the reader's convenience we employ a shorthand notation. Firstly, elements at a permutation's end and in their natural position are not shown. As a further simplification, we will often choose to denote a long sequence of symbols that remain unchanged and unused by prefix reversals, called a *block*, by a single signed element (with a $+$ or $-$ sign). The sign changes from $+$ to $-$, or conversely from $-$ to $+$, each time the element representing the block is part of a reversed prefix. The signed element in this way denotes the orientation of a block, i.e. whether it is reversed (denoted by $-$) or is in the original order (denoted by $+$). Blocks are a useful simplification for substrings of symbols that do not participate (i.e. do not come to the front) in some series of flips. In addition, we define a *clan* to be a consecutive sequence of integers (either in ascending or descending order).

Observation 1. For the permutation $\boldsymbol{\alpha} = \mathbf{2, 10, 5, 6, 7, 8, 4, 1, 3, 9}$, the sequence run($\alpha$) ends with the identity permutation.

Proof. Here is run(α) (with notational simplifications):

$$\mathbf{2, 10, 5, 6, 7, 8, 4, 1, 3, 9} \models\!= \mathbf{10, 2, 5, 6, 7, 8, 4, 1, 3, 9} \models\!= \mathbf{9, 3, 1, 4, 8, 7, 6, 5, 2} \models\!=$$
$$\mathbf{2, 5, 6, 7, 8, 4, 1, 3} \models\!= \mathbf{5, 2, 6, 7, 8, 4, 1, 3} \models\!= \mathbf{8, 7, 6, 2, 5, 4, 1, 3} \models\!= \mathbf{3, 1, 4, 5, 2} \models\!=$$
$$\mathbf{4, 1, 3, 5, 2} \models\!= \mathbf{5, 3, 1, 4, 2} \models\!= \mathbf{2, 4, 1, 3} \models\!= \mathbf{4, 2, 1, 3} \models\!= \mathbf{3, 1, 2} \models\!= \mathbf{2, 1} \models\!= \textbf{identity} \quad \square$$

Observation 2. For the signed permutation $\boldsymbol{\beta} = \mathbf{12, 8, 9, +6, 5, 7, 1, 11, 13, 10, 4, 3, 2}$, with just the one signed symbol, namely $+6$, the last permutation in the sequence run($\boldsymbol{\beta}$) truncated to the first 28 steps is $\mathbf{2, +6, 4, 1, 3, 5}$ (in the abbreviated form).

Proof. Here are the elements of run($\boldsymbol{\beta}$) for 28 flips:

$$\mathbf{12, 8, 9, +6, 5, 7, 1, 11, 13, 10, 4, 3, 2} \models\!= \mathbf{3, 4, 10, 13, 11, 1, 7, 5, -6, 9, 8, 12, 2} \models\!=$$
$$\mathbf{10, 4, 3, 13, 11, 1, 7, 5, -6, 9, 8, 12, 2} \models\!= \mathbf{9, +6, 5, 7, 1, 11, 13, 3, 4, 10, 8, 12, 2} \models\!=$$
$$\mathbf{4, 3, 13, 11, 1, 7, 5, -6, 9, 10, 8, 12, 2} \models\!= \mathbf{11, 13, 3, 4, 1, 7, 5, -6, 9, 10, 8, 12, 2} \models\!=$$
$$\mathbf{8, 10, 9, +6, 5, 7, 1, 4, 3, 13, 11, 12, 2} \models\!= \mathbf{4, 1, 7, 5, -6, 9, 10, 8, 3, 13, 11, 12, 2} \models\!=$$
$$\mathbf{5, 7, 1, 4, -6, 9, 10, 8, 3, 13, 11, 12, 2} \models\!= \mathbf{+6, 4, 1, 7, 5, 9, 10, 8, 3, 13, 11, 12, 2} \models\!=$$
$$\mathbf{9, 5, 7, 1, 4, -6, 10, 8, 3, 13, 11, 12, 2} \models\!= \mathbf{3, 8, 10, +6, 4, 1, 7, 5, 9, 13, 11, 12, 2} \models\!=$$
$$\mathbf{10, 8, 3, +6, 4, 1, 7, 5, 9, 13, 11, 12, 2} \models\!= \mathbf{13, 9, 5, 7, 1, 4, -6, 3, 8, 10, 11, 12, 2} \models\!=$$
$$\mathbf{2, 12, 11, 10, 8, 3, +6, 4, 1, 7, 5, 9} \models\!= \mathbf{12, 2, 11, 10, 8, 3, +6, 4, 1, 7, 5, 9} \models\!=$$
$$\mathbf{9, 5, 7, 1, 4, -6, 3, 8, 10, 11, 2} \models\!= \mathbf{10, 8, 3, +6, 4, 1, 7, 5, 9, 11, 2} \models\!=$$
$$\mathbf{11, 9, 5, 7, 1, 4, -6, 3, 8, 10, 2} \models\!= \mathbf{2, 10, 8, 3, +6, 4, 1, 7, 5, 9} \models\!=$$
$$\mathbf{10, 2, 8, 3, +6, 4, 1, 7, 5, 9} \models\!= \mathbf{9, 5, 7, 1, 4, -6, 3, 8, 2} \models\!= \mathbf{2, 8, 3, +6, 4, 1, 7, 5} \models\!=$$
$$\mathbf{8, 2, 3, +6, 4, 1, 7, 5} \models\!= \mathbf{5, 7, 1, 4, -6, 3, 2} \models\!= \mathbf{+6, 4, 1, 7, 5, 3, 2} \models\!=$$
$$\mathbf{3, 5, 7, 1, 4, -6, 2} \models\!= \mathbf{7, 5, 3, 1, 4, -6, 2} \models\!= \mathbf{2, +6, 4, 1, 3, 5}. \quad \square$$

Observation 3. For the signed permutation $\boldsymbol{\gamma} = \mathbf{2, 14, 9, 8, 7, +6, 5, 10, 11, 12, 4, 1, 3, 13}$, with just the one signed symbol, namely $+6$, the last permutation in the sequence run($\boldsymbol{\gamma}$) truncated to the first 17 steps is 2, $+6$, 4, 1, 3, 5 (in the abbreviated form).

Proof. Here are the elements of run($\boldsymbol{\gamma}$) for 17 flips:

$$\mathbf{2, 14, 9, 8, 7, +6, 5, 10, 11, 12, 4, 1, 3, 13} \models\!= \mathbf{14, 2, 9, 8, 7, +6, 5, 10, 11, 12, 4, 1, 3, 13}$$
$$\models\!= \mathbf{13, 3, 1, 4, 12, 11, 10, 5, -6, 7, 8, 9, 2} \models\!= \mathbf{2, 9, 8, 7, +6, 5, 10, 11, 12, 4, 1, 3}$$
$$\models\!= \mathbf{9, 2, 8, 7, +6, 5, 10, 11, 12, 4, 1, 3} \models\!= \mathbf{12, 11, 10, 5, -6, 7, 8, 2, 9, 4, 1, 3}$$
$$\models\!= \mathbf{3, 1, 4, 9, 2, 8, 7, +6, 5} \models\!= \mathbf{4, 1, 3, 9, 2, 8, 7, +6, 5} \models\!= \mathbf{9, 3, 1, 4, 2, 8, 7, +6, 5}$$
$$\models\!= \mathbf{5, -6, 7, 8, 2, 4, 1, 3} \models\!= \mathbf{2, 8, 7, +6, 5, 4, 1, 3} \models\!= \mathbf{8, 2, 7, +6, 5, 4, 1, 3}$$
$$\models\!= \mathbf{3, 1, 4, 5, -6, 7, 2} \models\!= \mathbf{4, 1, 3, 5, -6, 7, 2}| == \mathbf{5, 3, 1, 4, -6, 7, 2}$$
$$\models\!= \mathbf{+6, 4, 1, 3, 5, 7, 2} \models\!= \mathbf{7, 5, 3, 1, 4, -6, 2} \models\!= \mathbf{2, +6, 4, 1, 3, 5}. \quad \square$$

We are now ready to give the proof of Lemma 2, which we state again here for the convenience of the reader.

Lemma 2. *For all* $n \geq 18$, *such that* $n \equiv 2 (\text{mod } 8)$, run($\sigma_n$) *ends with the identity permutation.*

Proof of Lemma 2. As observed in the proof of Lemma 1, the first two flips for σ_n are of sizes n and $n-4$, respectively, resulting in the permutation $[\mathbf{5, 6, 7, 8, 9, \ldots, n-8}], \mathbf{n-5, n-6, n-2, n-7, 1, n-3, n-1, n-4, 4, 3, 2, n}$. The first $n-12$ symbols of this permutation are $n-12$ consecutive integers, namely the sequence $5, 6, 7, 8, 9, \ldots, (n-8)$.

The next flip is a flip of size 5 which will bring 9 to the front, creating the clan [9, 8, 7, 6, 5], followed by a flip of size 9 which will bring 13 to the front, making two clans in front, namely [13, 12, 11, 10] followed by [5, 6, 7, 8, 9], then a flip of size 13 will bring 17 to the front, making three clans, namely [17, 16, 15, 14], [9, 8, 7, 6, 5], followed by [10, 11, 12, 13].

In fact, the next sequence of flips is 5, 9, 13, 17, ..., i.e. $\{5 + 4j\}$, for $0 \le j \le (n - 14)/4$. The first flip of size 5 creates a clan of size 5, as noted. Each subsequent flip creates a clan of size 4, but these successive clans are put at opposing ends of this sequence of clans, due to the nature of prefix reversals. Specifically, at the end of the flip sequence 5, 9, 13, 17, ..., i.e. $\{5 + 4j\}$, for $0 \le j \le (n - 14)/4$, the permutation will be changed to

$$\mathbf{n-2}, \mathbf{n-6}, \mathbf{n-5}, \mathbf{n-8}, [\mathbf{n-13}, \mathbf{n-14}, \mathbf{n-15}, \mathbf{n-16}], \dots, [\mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}], [\mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}], \dots,$$
$$[\mathbf{n-12}, \mathbf{n-11}, \mathbf{n-10}, \mathbf{n-9}], \mathbf{n-7}, \mathbf{1}, \mathbf{n-3}, \mathbf{n-1}, \mathbf{n-4}, \mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{n}.$$

We now view $[\mathbf{n-13}, \mathbf{n-14}, \mathbf{n-15}, \mathbf{n-16}], \dots, [\mathbf{n-12}, \mathbf{n-11}, \mathbf{n-10}]$ as a large block, temporarily eliminate it, and keep track of its orientation by using the signed number $n - 8$ just before it, a shorthand notational convenience that we described above. The reason for the elimination is that none of its elements come to the front in the next few steps and the block remains intact with the same orientation, in postions following the element $n - 8$ at the end of these steps. The elimination is thus a simplification for the ease of the reader. To make the shortened string a permutation we subtract $n - 14$ from each number greater than $n - 13$, and simplify further by eliminating the number n from the end. The result is the abbreviated signed permutation $\mathbf{12}, \mathbf{8}, \mathbf{9}, +\mathbf{6}, \mathbf{5}, \mathbf{7}, \mathbf{1}, \mathbf{11}, \mathbf{10}, \mathbf{4}, \mathbf{3}, \mathbf{2}$. By Observation 2, after 28 flips one obtains the reduced signed permutation $\mathbf{2}, +\mathbf{6}, \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{5}$. Putting back the temporarily eliminated large block that was represented by the signed number $+6$, and restoring the original numbers, we get

$$\mathbf{2}, \mathbf{n-8}, [\mathbf{n-13}, \mathbf{n-14}, \mathbf{n-15}, \mathbf{n-16}], [\mathbf{n-21}, \mathbf{n-22}, \mathbf{n-23}, \mathbf{n-24}], \dots,$$
$$[\mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}], [\mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}], \dots,$$
$$[\mathbf{n-20}, \mathbf{n-19}, \mathbf{n-18}, \mathbf{n-17}], [\mathbf{n-12}, \mathbf{n-11}, \mathbf{n-10}], \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{n-9}.$$

So, this is the permutation reached after 28 flips, with the numbers $n - 7, \dots, n$ in the correct order at the end and not included.

Now simplifying again, but in a different way, choose the large block $[\mathbf{n-21}, \mathbf{n-22}, \mathbf{n-23}, \mathbf{n-24}], \dots [\mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}]$, $[\mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}], \dots, [\mathbf{n-20}, \mathbf{n-19}, \mathbf{n-18}]$ and temporarily eliminate it. We keep track of this block and its orientation using the signed number $n - 16$ immediately in front of the block. To make the resulting string into a permutation, we subtract $n - 22$ from each symbol greater than $n - 21$. The result is the reduced signed permutation

$$\mathbf{2}, \mathbf{14}, \mathbf{9}, \mathbf{8}, \mathbf{7}, +\mathbf{6}, \mathbf{5}, \mathbf{10}, \mathbf{11}, \mathbf{12}, \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{13}.$$

By Observation 3 after 17 flips, one obtains the reduced signed permutation $\mathbf{2}, +\mathbf{6}, \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{5}$, matching the one we obtained at the end of the flip sequence indicated in Observation 1. Putting back the large block that the signed number $+6$ represented, and restoring the original symbols, we obtain

$$\mathbf{2}, \mathbf{n-16}, [\mathbf{n-21}, \mathbf{n-22}, \mathbf{n-23}, \mathbf{n-24}], \dots, [\mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}], [\mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}], \dots,$$
$$[\mathbf{n-20}, \mathbf{n-19}, \mathbf{n-18}], \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{n-17}$$

where the symbols $\mathbf{n-15}, \mathbf{n-14}, \dots, \mathbf{n}$ are in order at the end of the permutation and are not shown.

Now choose the large block $[\mathbf{n-29}, \mathbf{n-30}, \mathbf{n-31}, \mathbf{n-32}], \dots [\mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}], [\mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}], \dots, [\mathbf{n-28}, \mathbf{n-27}, \mathbf{n-26}]$ to be represented by the signed number $n-24$, and subtract $n-30$ from each element in the resulting string greater than $n - 29$. The result is the reduced signed permutation

$$\mathbf{2}, \mathbf{14}, \mathbf{9}, \mathbf{8}, \mathbf{7}, +\mathbf{6}, \mathbf{5}, \mathbf{10}, \mathbf{11}, \mathbf{12}, \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{13},$$

which is the same as we had before.

So, the 17 flips described in Observation 3 are used again and result in 8 more largest numbers being put at the end of the permutation into sorted order and hence never moved again. In this way, after $(n - 18)/8$ iterations of the 17 flip sequence given in Observation 3, we obtain the permutation $\mathbf{2}, \mathbf{10}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{4}, \mathbf{1}, \mathbf{3}, \mathbf{9}$, with all other numbers in order at the end. By Observation 1, we get the identity from this permutation. □

3. Conclusions

The permutation family $\{\sigma_n | n > 25 \text{ with } n \equiv 2 (\text{mod } 8)\}$ in $\Pi^{(8)}$ we described is not the best family of permutations we discovered. It was chosen, because its run sequence is relatively simple and hence makes easier the proof of termination to the identity permutation. In fact, we know of other families of permutations that also go to the identity permutation, which provide a better constant for the $\Omega(n^2)$ lower bound. For example, the family of permutations $\{\gamma_n | n > 20 \text{ with } n \equiv 9 (\text{mod } 12)\}$ in $\Pi^{(12)}$ defined by

$$\gamma_{\mathbf{n}} = \mathbf{n}, (\mathbf{2}, \mathbf{3}, \dots, \mathbf{n-12}), \mathbf{n-10}, \mathbf{n-1}, \mathbf{n-9}, \mathbf{n-4}, \mathbf{n-8}, \mathbf{n-2}, \mathbf{n-7}, \mathbf{n-11}, \mathbf{1}, \mathbf{n-5}, \mathbf{n-3}, \mathbf{n-6}$$

is such that $\text{run}(\gamma_n)$ has length $(21n - 77)/4$ (see [10]) and ends with the identity permutation. By chaining permutations in the family $\{\gamma_n\}$ we achieve an overall run length of $\frac{7}{96}\left(3n^2 + 14n - 369\right)$. By contrast, the described family $\{\sigma_n | n > 25$ with $n \equiv 2 (\text{mod } 8)\}$ in $\Pi^{(8)}$ is such that $\text{run}(\sigma_n)$ has length $17(n + 30)/8$. We also discovered a family of permutations

$\{\tau_n | n > 68 \text{ with } n \equiv 5 (\text{mod } 8)\}$ in $\Pi^{(11)}$, defined by

$$\boldsymbol{\tau_n = n-4, (2, 3, \ldots, n-11), n-9, n-3, n-10, n, n-5, n-7, n-8, n-1, 1, n-6, n-2}$$

and proved that run(τ_n) has length greater than $(n^2 - 55n + 1294)/12$ [10]. However, this run sequence does not end with the identity and hence cannot be used in chaining.

Nothing we have seen suggests the existence of permutations whose iterate sequences have length asymptotically larger than n^2, except possibly the family $\{\tau_n\}$ in $\Pi^{(11)}$, which, if they were to end with the identity permutation, hence make chaining possible, would yield an $\Omega(n^3)$ lower bound. In addition, substantially improving Knuth's $F_{n+1} - 1$ upper bound ([9], p. 119) seems to be very challenging.

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